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Crack propagation in a dynamic fuse model of electromigration

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Abstract. We introduce a dynamic fuse model for the damage done to a metal thin film by electromigration, and study the growth of a single crack that is perpendicular to the direction of the ambient current. As the crack length $2x$ grows large, the velocity of the crack tips v scales as $v(x) \sim x^{\alpha}$. We argue that the value of the exponent α is exactly 2. This result is in excellent agreement with our numerical work.

There is currently much interest in random media that are changed irreversibly by an applied field. These so-called 'breakdown problems' play an important role in non-equilibrium statistical physics and materials science. 'Burn out' of random fuse networks [1–5], dielectric breakdown [2, 5–12], the onset of superconductivity in granular superconductors [13–17], and the fracture of brittle materials [18–25] have all been studied using breakdown models.

Most of the progress on breakdown models has come from Monte Carlo simulations. However, in an important series of papers, Duxbury and co-workers developed a 'Lifshitz-type' theory for the breakdown voltage of a two-dimensional random fuse network in the limit in which the insulating bonds are initially dilute [2–4]. The first step in constructing this theory was to determine the breakdown voltage of a line defect (or 'crack') oriented perpendicularly to the direction of the ambient current.

The breakdown models mentioned so far are quasi-static, since failure occurs instantaneously when the applied voltage (or stress) is sufficiently large. The first truly kinetic breakdown model was introduced by Sornette and Vanneste [26, 27]. Their model describes the failure of fuse networks that burn out due to Joule heating. In their model, the temperature T of a fuse with resistance R carrying current I obeys the equation

$$dT/dt = R|I|^b - aT \tag{1}$$

where a and b are non-negative constants. The term $R|I|^b$ accounts for a generalized Joule heating of the fuse; for real fuses $b = 2$. The second term on the right-hand side of (1) is the rate that heat is lost to the substrate. When the temperature of a fuse reaches a given threshold, it burns out irreversibly and becomes an insulator. The Monte Carlo simulations of Vanneste and Sornette revealed a rich phenomenology of fracture patterns and the existence of a novel dynamical memory effect [26, 27].

There are many failure processes in nature, and we expect that all will display time-dependent effects under certain circumstances. For example, when an electrical current passes through a thin metal film, collisions between the conduction electrons and the metal ions lead to drift of the ions. This process is known as electromigration [28–30]. If there

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is a divergence in the flux of ions at a point, a void or hillock forms [31]. Voids grow and overlap until conduction ceases and electrical failure is complete. Electromigration can lead to the electrical failure of interconnects in VLSI circuits in relatively short times, reducing the circuit lifetime to an unacceptable level [32]. It is therefore of great technological importance to understand and control electromigration failure of thin films. Electromigration-induced damage in a polycrystalline metal film is an irreversible kinetic process, since the damage cannot be repaired simply by reversing the current.

In this paper, we introduce a kinetic breakdown model of electromigration failure. As a first step towards understanding the failure process when many cracks are present, we study the growth of a single crack that is perpendicular to the direction of the ambient current. As the crack length $2x$ grows large, the velocity of the crack tips v scales as $v(x) \sim x^{\alpha}$. We argue that the value of the exponent α is *exactly* 2. This result is in excellent agreement with our numerical work, and is the first analytical result obtained on a dynamic fuse model.

We adopt a simple coarse-grained description of a disordered polycrystalline metal film. Consider a regular square grid of sites in which each nearest-neighbour pair of sites is joined by either a conducting wire with resistance R (with probability p) or by an insulator (with probability $1-p$). This percolative disorder is meant to mimic the disordered crystal structure in a real polycrystalline metal film.

When a current passes through a particular wire in the grid, electromigration occurs and electrical failure eventually takes place. The current I passing through this wire may vary with time since failures elsewhere in the system lead to current redistribution. It is natural to assume that the rate damage is done to the wire at time t is proportional to $|I(t)|$. We further assume that once the damage done to the wire has reached a given threshold, the wire fails irreversibly and becomes an insulator. The lifetime of the wire t_f is therefore given by

$$\int_0^{t_f} |I(t)|\,\mathrm{d}t = Q_0\,. \tag{2}$$

Equation (2) states that once a charge Q_0 has flowed through the wire, it fails. The absolute value of the current appears in the failure criterion (2) because electromigration damage cannot be repaired simply by reversing the current. Indeed, ac currents lead to electromigration failure in times comparable to DC currents of the same magnitude.

Clearly, for $a = 0$ and $b = 1$, the more general failure criterion of Sornette and Vanneste reduces to ours if the temperature T is replaced by the charge $Q(t) = \int_0^t |I(t')|\,\mathrm{d}t'$. Note, however, that Sornette and Vanneste did not actually study this case. As we shall see, it is possible to make significant progress analytically on this special case, and so it is of particular interest.

The behaviour of our model for general values of p is complex. As a simple starting point, we study the failure of a network with $p = 0$.

Let the lattice spacing of the square grid be a. We take the coordinates (x, y) of the sites in the grid to be $(n_x a, (n_y + \frac{1}{2})a)$, where n_x and n_y are integers. Each bond in the lattice is a resistor with conductance σ. We place busbars on the rows at $y = +\infty$ and $y = -\infty$. A constant voltage difference is applied across these two busbars starting at time $t = 0$, so that the vertical component of the electric field has the constant value $\Delta V/a$ up until the time the network fails.

Clearly, in this case all of the horizontal bonds in the grid carry zero current, while each of the vertical bonds carries current $\sigma \Delta V$ until failure occurs. All of the vertical bonds fail simultaneously at time $t = Q_0/(\sigma \Delta V)$.

Now consider the next level of complexity. Suppose that at time $t = 0$ there is a single horizontal 'crack' of length $2l_0a$, where l_0 is a non-negative integer. Specifically, we assume that initially the vertical bonds whose centres are at $(na, 0)$ with $n = -l_0, -l_0 + 1, \ldots, l_0 + 1, l_0$ are broken, and that all of the other bonds in the network are conducting at $t = 0$.

Far from the crack, the current distribution is unchanged. However, the current flowing through bonds close to the crack tips is increased dramatically by the presence of the crack. This current enhancement is largest in the two unbroken vertical resistors adjacent to the crack tips, and as a result, these are the first bonds to fail. The subsequent behaviour of the crack is less obvious, since the failure process in our model is a cumulative effect. Our simulations show that at all times, the next bonds to fail are the vertical bonds immediately adjacent to the crack tips. Thus, the crack tips propagate laterally until network failure is complete. The vertical bonds with their centres at heights $y \neq 0$ and all of the horizontal bonds remain conducting throughout the failure process. Our simulations strongly suggest that these observations apply for all values of l_0.

We can readily write down an equation of motion for the crack tips. Suppose that the crack length is $2la$ at a given time, and let $I(n, l)$ denote the current flowing through the vertical bond with its centre at $(na, 0)$ at this time. Further, let t_n be the time when this bond fails, and adopt the convention that $t_{l_0} = 0$. Finally, we set $\Delta t_n = t_{n+1} - t_n$ for $n \geqslant l_0$. The equation of motion is

$$\sum_{l=l_0}^{n-1} I(n, l)\Delta t_l = Q_0 \tag{3}$$

where $n > l_0$. Equation (3) simply states that the bond with its centre at $(na, 0)$ fails once the net amount of charge that has flowed through it is Q_0. Note that the direction of the current through the vertical bonds never changes, and so it is not necessary to write $|I(n, l)|$ in lieu of $I(n, l)$ in (3).

If the initial length of the crack is large, the discrete lattice structure will have little effect on the crack dynamics. In the same way, if the initial length of the crack is comparable to the lattice spacing, the lattice structure will become unimportant once the crack has grown to sufficient size. In both of these circumstances, a continuum approximation to the equation of motion (3) may be applied.

In the continuum limit, the lattice spacing a tends to zero. The current density far from the crack $j_0 \equiv \sigma\Delta V/a$, the initial length of the crack $x_0 \equiv 2l_0a$, and $q_0 \equiv Q_0/a$ all tend to constants in this limit. Equation (3) becomes

$$\int_0^t j_y(x(t), x(t'))\,\mathrm{d}t' = q_0\,. \tag{4}$$

Here $2x(t)$ is the length of the crack at time t and $j_y(x, x')$ is the vertical component of the current density at $(x, 0)$ when the crack has length $2x'$. The current density $j_y(x, x')$ is [3]

$$j_y(x, x') = j_0\left[1 - \left(\frac{x'}{x}\right)^2\right]^{-1/2}. \tag{5}$$

Let $v(x, x_0)$ be the speed of the crack tip when the crack's length is $2x$. We will now use (4) to determine the behaviour of $v(x, x_0)$ for $x \gg x_0$. Introducing the new variable of integration $x' \equiv x(t')$ in (4), we have

$$\int_{x_0}^{x} j_y(x, x')[v(x', x_0)]^{-1}\,\mathrm{d}x' = q_0\,. \tag{6}$$

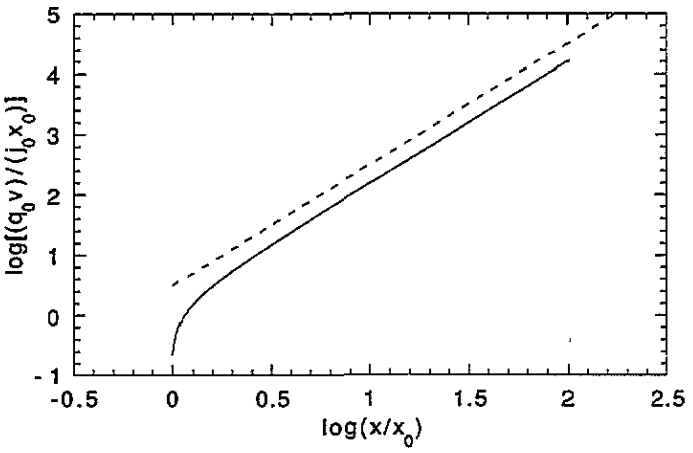

Figure 1. $\log[q_0 v(x, x_0)/(j_0 x_0)]$ plotted versus $\log(x/x_0)$ (full curve). The broken line has a slope of 2.

Since x_0 is the only length scale in the problem, it makes sense to introduce the dimensionless widths $w \equiv x/x_0$ and $w' \equiv x'/x_0$. Applying these definitions and equation (5) in (6), we obtain

$$\int_1^w \frac{x_0\,\mathrm{d}w'}{v(x_0 w', x_0)\sqrt{1-(w'/w)^2}} = \frac{q_0}{j_0}. \tag{7}$$

The quantity $q_0 v(x, x_0)/(j_0 x_0)$ is a dimensionless function of x and x_0. Since x_0 is the only length scale remaining in the continuum limit, $q_0 v(x, x_0)/(j_0 x_0)$ cannot depend on x and x_0 separately—it can only depend on their ratio x/x_0. Thus, we have the scaling form

$$v(x, x_0) = \frac{j_0 x_0}{q_0} f\left(\frac{x}{x_0}\right) \tag{8}$$

where f is a dimensionless function of x/x_0.

To learn something of the scaling function f, we solved (6) numerically using a finite-difference approximation. In figure 1, $\log[q_0 v(x, x_0)/(j_0 x_0)]$ is plotted against $\log(x/x_0)$. The curve rapidly becomes linear as x/x_0 is increased, showing that

$$f(w) \sim K w^{\alpha} \tag{9}$$

for $w \gg 1$. Here K is a finite, non-zero constant independent of both x and x_0. Figure 1 also shows that the exponent α is close to 2. A linear least-squares fit to the curve for $x/x_0 \geqslant 10$ yields $\alpha = 2.001\,20 \pm 0.000\,01$. The error quoted here does not take into account the fact that the slope of the curve slowly decreases with $\log(x/x_0)$. If the curve is fit for $x/x_0 \geqslant 20$, for example, a value of α still closer to 2 results.

Inspired by these numerical results, we assume that (9) is valid, and that $1 < \alpha < 3$. We shall now demonstrate that once these assumptions have been made, it follows that α must be *exactly* equal to 2.

Let $z = w'/w$. Using (8), we can rewrite (7) as follows:

$$\int_{1/w}^1 \frac{w\,\mathrm{d}z}{f(wz)\sqrt{1-z^2}} = 1. \tag{10}$$

We cannot simply replace $f(wz)$ in this integral by $Kw^{\alpha}z^{\alpha}$ because wz is not large throughout the range of integration. Corrections to the asymptotic scaling form (9) must therefore be taken into account. By assumption, $w^{\alpha}[f(w)]^{-1}$ tends to a finite constant K^{-1} as w tends to infinity. For large but finite w, there will be corrections to this leading-order behaviour. These corrections will become increasingly important as w is decreased. It is therefore natural to assume that

$$w^{\alpha}[f(w)]^{-1} = \sum_{n=0}^{\infty} A_n w^{-n} \tag{11}$$

for all $w \geqslant 1$. Note that $A_0 = 1/K$. Inserting (11) into (10), we have

$$w^{1-\alpha} \sum_{n=0}^{\infty} A_n w^{-n} \int_{1/w}^{1} \frac{\mathrm{d}z}{z^{\alpha+n}\sqrt{1-z^2}} = 1 . \tag{12}$$

To proceed further, we must analyse the integrals that appear on the left-hand side of (12). Using our assumption that $1 < \alpha < 3$, we obtain

$$\int_{1/w}^{1} \frac{\mathrm{d}z}{z^{\alpha+n}\sqrt{1-z^2}} = \frac{w^{\alpha+n-1}}{\alpha+n-1} + C_n(\alpha) + \mathrm{O}(w^{\alpha+n-3}) \tag{13}$$

for $w \gg 1$. Here

$$C_n(\alpha) \equiv \int_0^1 \left[\frac{1}{\sqrt{1-z^2}} - \sum_{l=0}^{\lceil n/2 \rceil} (-1)^l \binom{-\frac{1}{2}}{l} z^{2l}\right] \frac{\mathrm{d}z}{z^{\alpha+n}} + \sum_{l=0}^{\lceil n/2 \rceil} (-1)^l \binom{-\frac{1}{2}}{l} \frac{1}{2l+1-\alpha-n} (1 - \delta_{2l+1-\alpha-n,0}) \tag{14}$$

is a constant. In (14), $\lceil x \rceil$ denotes the smallest integer greater than or equal to x and the Kronecker delta $\delta_{m,n}$ is 1 if $m = n$ and is zero otherwise. Note that for the special case $\alpha + n = 3$, the correction term of order $w^{\alpha+n-3}$ in (13) must be replaced by a term of order $\log w$.

Substituting the asymptotic form (13) in (12), we find that

$$\sum_{n=0}^{\infty} \frac{A_n}{\alpha+n-1} + A_0 C_0(\alpha) w^{1-\alpha} + \mathrm{O}(w^{-\beta}) = 1 \tag{15}$$

for $w \gg 1$. Here β is the smaller of α and 2. We see that we must have

$$\sum_{n=0}^{\infty} \frac{A_n}{\alpha+n-1} = 1 \tag{16}$$

and that

$$C_0(\alpha) = 0 . \tag{17}$$

Equation (17) uniquely specifies the value of α, as we shall now demonstrate. Explicitly,

$$C_0(\alpha) \equiv \int_0^1 \left[\frac{1}{\sqrt{1-z^2}} - 1\right] \frac{\mathrm{d}z}{z^{\alpha}} - \frac{1}{\alpha-1} .$$

$C_0(\alpha)$ is an increasing function of α for $1 < \alpha < 3$, and $C_0(2) = 0$. Thus, (17) has a single root on the interval $1 < \alpha < 3$, and this root occurs at $\alpha = 2$. We conclude that α is exactly 2, and that for $x \gg x_0$,

$$v(x, x_0) \cong \frac{K j_0}{q_0 x_0} x^2 . \tag{18}$$

As we have seen, our result $\alpha = 2$ is in excellent agreement with the results of our numerical integration of the continuum equation of motion. We have also used the exact equation of motion (3) to find the tip velocity, and have verified that it agrees with our result (18) in the asymptotic limit [33].

In our model, the crack tips accelerate as the length of the crack grows. Initially, the crack tips do not move as damage accumulates in the network. The crack tips move with increasing speed after the first bonds are broken. This acceleration occurs because the current crowding at the crack tips becomes more marked with the passage of time, and because the damage to the bonds is cumulative.

We handled the lower limit of integration with special care in our analysis of our equation of motion (6). Care was needed because when $x \gg x_0$, the dominant contribution to the integral comes from x' close to x_0. This can be understood both from a mathematical and from a physical standpoint. We begin with the formal demonstration. The integrand in (13) has a non-integrable singularity at $z = 0$. As a result, the dominant contribution to this integral comes from z close to $1/w$, that is, from x' close to x_0. The same is also true of the equation of motion (10). We now turn to our heuristic explanation. Because the crack tips move slowly at first and then accelerate, most of the charge needed to break a bond flows through it while the crack tip is still far away. We again conclude that the dominant contribution to the integral on the left-hand side of (6) comes from x' relatively close to x_0.

Now that we know the asymptotic behaviour of $v(x, x_0)$, we can integrate to find the time dependence of the tip location x for $x \gg x_0$. Clearly,

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x'}{v(x', x_0)} . \tag{19}$$

Inserting (8) and (11) in (19) and using the result $\alpha = 2$, we obtain

$$\sum_{n=0}^{\infty} \frac{A_n}{n+1} \left[1 - \left(\frac{x_0}{x} \right)^{n+1} \right] = \frac{j_0 t}{q_0} .$$

For $x \gg x_0$, this becomes

$$1 - A_0 \left(\frac{x_0}{x} \right) \cong \frac{j_0 t}{q_0}$$

where we have employed (16) and the fact that $\alpha = 2$. Finally, we have the desired result: for $x \gg x_0$

$$x \cong K^{-1} \frac{x_0}{1 - j_0 t / q_0} . \tag{20}$$

Interestingly, (20) shows that the time to failure in the presence of the crack is q_0/j_0. This is the time it takes a defect-free film to fail, and so the presence of the crack does not reduce the lifetime of an infinitely large film. For a finite-size film, of course, the lifetime will

always be reduced by the presence of a crack. What we have shown is that this reduction tends to zero in the limit in which the film dimensions become large compared with the initial length of the crack.

When p is small but non-zero, there are multiple cracks present in our model. The interactions between these cracks will be negligible at early times, and each crack will have a tip velocity $v(x)$ that grows as x^2. At later times, crack–crack interactions become increasingly important, and cracks begin to coalesce as well as to grow. In future publications, we will develop a theory for the late stages of the electromigation process, and compare the results of this theory with our Monte Carlo simulations [33]. We are also exploring the possibility that our theory can be extended to yield the asymptotic behaviour of a crack in the dynamical thermal fuse model with arbitrary values of a and b.

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References

[1] de Arcangelis L, Redner S and Herrmann H J 1985 *J. Physique Lett.* **46** L585
[2] Duxbury P M, Beale P D and Leath P L 1986 *Phys. Rev. Lett.* **57** 1052
[3] Duxbury P M, Leath P L and Beale P D 1987 *Phys. Rev.* B **36** 367
[4] Duxbury P M and Leath P L 1987 *J. Phys. A: Math. Gen.* **20** L411
[5] Söderberg M 1987 *Phys. Rev.* B **35** 352
[6] Takayasu H 1985 *Phys. Rev. Lett.* **54** 1099
[7] Chakrabarti B K, Bardhan K K and Ray P 1987 *J. Phys. C: Solid State Phys.* **20** L57
[8] Manna S S and Chakrabarti B K 1987 *Phys. Rev.* B **36** 4078
[9] Beale P D and Duxbury P M 1988 *Phys. Rev.* B **37** 2785
[10] Bowman D R and Stroud D 1989 *Phys. Rev.* B **40** 4641
[11] Gyure M F and Beale P D 1989 *Phys. Rev.* B **40** 9533
[12] Gyure M F and Beale P D 1992 *Phys. Rev.* B **46** 3736
[13] Bradley R M, Kung D, Doniach S and Strenski P N 1987 *J. Phys. A: Math. Gen.* **20** L911
[14] Bradley R M, Kung D, Strenski P N and Doniach S 1988 *Physica* **152B** 282
[15] Leath P L and Tang W 1989 *Phys. Rev.* B **39** 6485
[16] Xia W and Leath P L 1989 *Phys. Rev. Lett.* **63** 1428
[17] Hinrichsen E L, Roux S and Hansen A 1990 *Physica* **167C** 433
[18] Ray P and Chakrabarti B K 1985 *Solid State Commun.* **53** 477
[19] Ray P and Chakrabarti B K 1985 *J. Phys. C: Solid State Phys.* **18** L185
[20] Chakrabarti B K, Chowdhury D and Stauffer D 1986 *Z. Phys.* B **62** 343
[21] Sahimi M and Goddard J D 1986 *Phys. Rev.* B **33** 7848
[22] Beale P D and Srolovitz D J 1988 *Phys. Rev.* B **37** 5500
[23] Hansen A, Roux S and Herrmann H J 1989 *J. Physique* **50** 733
[24] Hassold G N and Srolovitz D J 1989 *Phys. Rev.* B **39** 9273
[25] Herrmann H J, Hansen A and Roux S 1989 *Phys. Rev.* B **39** 637
[26] Sornette D and Vanneste C 1992 *Phys. Rev. Lett.* **68** 612
[27] Vanneste C and Sornette D 1992 *J. Physique* I **2** 1621
[28] Huntington H B 1975 *Diffusion in Solids—Recent Developments* ed A S Nowick and J J Burton (New York: Academic)
[29] Pratt J N and Sellors R G R 1973 *Electrotransport in Metals and Alloys* (Riehen: Trans Tech)
[30] Rigney D A 1974 *Charge Transfer—Electronic Structure of Alloys* ed L H Bennett L H and R H Willens (New York: AIME)
[31] d'Heurle F M and Ho P S 1978 *Thin Films—Interdiffusion and Reactions* ed J M Poate, K N Tu and J W Mayer (New York: Wiley)
[32] Ghate P B 1983 *Solid State Technol.* March issue p 113
[33] Wu K and Bradley R M 1993 unpublished